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Analysis of Trunk Groups Containing Short-Holding-Time Trunks

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This paper presents models for the behavior of trunk groups containing short-holding-time faulty trunks. The models, referred to as ordered selection, two-sided selection, random selection, or queuing selection, are applicable to selection procedures used by a number of switching systems. Each model is analyzed to obtain the fraction of group attempts carried on a faulty trunk in the group, and the corresponding fraction of group attempts that find all trunks busy (blocking or overflow). Numerical results for the basic models are also presented. The results indicate that factors such as the trunk selection procedure or the type of group (high usage or final) can lead to significant differences in the performance of a group containing a faulty trunk.

I. INTRODUCTION

A message trunk, the basic connecting link in the switched telephone network, provides the supervising, signaling, and ringing capabilities essential to call set-up, as well as the communication path. When a condition that prevents proper functioning of a message trunk occurs, and causes a call failure, the trunk is normally released by a switching system on customer abandonment of the failed attempt and is then available to fail another call. As a result, a single undetected faulty trunk can fail a disproportionate fraction of the offered attempts to a group. Figure 9 shows an illustrative case where one trunk, subsequently verified to have been faulty, carried 35 out of 77 attempts offered to a trunk group during one hour. Because of their potential service impact, such trunks are of major concern throughout the Bell System and the object of many preventive maintenance and trouble detection programs.

This paper presents models and analyses for the behavior of trunk groups containing short-holding-time trunks. This terminology is used to emphasize that the faulty trunks of interest are accessible by the customer, as opposed to faulty trunks that inhibit seizure by a switching system (false busys) or that result in an automatic retrial on

another trunk when an abnormal condition is detected. The models were developed to provide tools for quantifying the potential impact of short-holding-time trunks, to help determine more effective ways to use the trouble detection resources in the network. Although the main emphasis here is on the analytical development of the models, some numerical results are also presented. The reader primarily interested in these results can refer directly to Section III. (Necessary terminology and the relationship of the idealized models to switching systems are summarized in Section 2.1 and Table I, respectively.) A more detailed outline follows.

The basic models studied, referred to as ordered selection, two-sided selection, random selection, and queuing selection, and the switching systems to which they primarily apply, are summarized in Table I. Justification for the choice of the models is given in Ref. 1. Basic assumptions are Poisson input, exponential holding times for both normal- and short-holding-time trunks,* and no retrials because of blocked or ineffective attempts.† Within this framework, the impact of a short-holding-time trunk depends on its position in the group and on the way idle trunks are selected. Impact is quantified in terms of the fraction ineffective (fraction of offered attempts carried on the short-holding-time trunk) and the blocking (fraction of offered attempts that find all trunks busy in the group). These measures enable total ineffectives for both high usage groups (fraction ineffective) and final groups (fraction ineffective plus blocking) to be determined, as well as permit the distortion in standard traffic measurements (group usage and counts of offered and overflow attempts) owing to a short-holding-time trunk to be assessed. It should be noted that blocking, used here in a conventional traffic engineering sense, is used by some to refer to call failures resulting from any cause.

In Section II, prior work is briefly discussed and analysis for the various models is developed. Terminology common to the models is summarized in Section 2.1. In Section 2.2, ordered selection is considered. An exact solution for the fraction ineffective is obtained. The computations for the special case of one short-holding-time trunk are summarized in (14) to (18). Blocking is treated in an approximate way, using the equivalent random method.[2]

For two-sided selection (Section 2.3), the fraction ineffective is approximated by using results for ordered selection. The computa-

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* Although our main interest is short-holding-time trunks, some results hold for a short or long holding time on one trunk. In these cases, the trunk is referred to as abnormal.

† The models, however, can be used with retrials, if these are also treated as Poisson (this approximation was found to be reasonable). However, the inclusion of retrials (which are an important factor in considering volume changes) does not substantially change the service impact of a short-holding-time trunk.

Table I — Summary of models for machine trunk selection

Trunk Selection Model	Description	Switching System*
Ordered selection	Fixed-order hunt for an idle trunk.	Step-by-step, Panel, No. 4 Crossbar
Two-sided selection	Traffic split into two parts. Each part uses a fixed-order hunt for an idle trunk, with the two orders reversed.	No. 1 Crossbar, Crossbar Tandem, No. 4 Crossbar (two-way)
Random selection	Equally likely choice of an idle trunk.	No. 5 Crossbar
Queuing selection	Trunk that has been idle the longest is chosen.	No. 1 Electronic switching system

* Groups are assumed one-way (trunks are selected from only one end of the group), except for No. 4 Crossbar (two-way), where a two-way group between No. 4 Crossbar systems is assumed. The applications are not always precise; trunk assignments to frames, gradings, and certain subgrouping arrangements can affect the selection procedure of some systems.

tions summarized in (25) to (37) give good results for the fraction ineffective. Blocking is also treated in an approximate way.

Random and queuing selection are considered in Section 2.4. It is shown that, for fraction ineffective and blocking, these are equivalent. Simple exact solutions involving the Erlang B formula are obtained. Equation (60) gives the fraction ineffective, and eq. (61) gives the blocking.

Numerical results are given in Section III. For reasonable values of normal- to short-holding-time ratios as suggested by field data (about 5 to 30, depending on the type of fault and type of traffic), the results confirm that a single short-holding-time trunk can have a severe impact on service. For example, Fig. 6 shows the impact of one short-holding-time trunk in a group with random/queuing selection. It is apparent that the short-holding-time trunk has a significant impact over a wide range of conditions.

For any group, assuming that a short-holding-time trunk is equally likely to be in any position,* the various disciplines can be compared. We find that (see Fig. 8)

$$\begin{bmatrix}\text{Average}\\ \text{fraction}\\ \text{ineffective}\\ \text{for}\\ \text{random or}\\ \text{queuing}\\ \text{selection}\end{bmatrix} \leqq \begin{bmatrix}\text{Average}\\ \text{fraction}\\ \text{ineffective}\\ \text{for}\\ \text{two-sided}\\ \text{selection}\end{bmatrix} \leqq \begin{bmatrix}\text{Average}\\ \text{fraction}\\ \text{ineffective}\\ \text{for}\\ \text{ordered}\\ \text{selection}\end{bmatrix},$$

* For random/queuing selection, the position is irrelevant.

i.e., the random/queuing selection tends to minimize the impact of a short-bolding-time trunk. Of course, for a fixed position for a sbort-holding-time trunk, either ordered or two-sided selection can have a significantly lower fraction ineffective than random/queuing (e.g., last trunk for ordered selection). These results suggest tbat, if expected service improvement is used as a criterion, then, all other things being equal, tbere is a higher service payoff in eliminating faults from trunk groups in older switching systems than in more modern systems. Conversely, the more modern systems provide better service for a given level of trunk faults.

Section 3.5 briefly discusses practical limitations to the model assumptions and gives further consideration to Fig. 9.

II. ANALYSIS MODELS FOR GROUPS CONTAINING SHORT-HOLDING-TIME TRUNKS

In this section, the equilibrium behavior of groups containing short-holding-time trunks is considered. Prior work in this area is limited. Klimontowicz[3] appears to be the first to give tbe problem attention, considering random selection of idle trunks, ordered selection, and cyclic random (sequential with initial starting point chosen randomly). He develops some analytic results for special cases such as zero holding time on the abnormal trunk, but relies on simulation for most of his results. In support of work to detect faulty trunks from operator trouble reports,[4] Forys[5] has considered ordered selection for a trunk group with mean bolding time dependent on trunk position. In this general case, the fraction of attempts carried on any trunk can be determined by applying renewal theory, but the results are numerically inconvenient. (For completeness, these results are included in Section 2.1.) However, ordered selection with a single abnormal trunk permits application of a known recursion formula from Ref. 6, to give the computationally convenient solution developed in Section 2.1.

The random selection model has also been considered in the context of fault detection from trouble reports.[7] Except for limiting cases, analytic solutions were not obtained in Ref. 7; numerical solutions of the state equations to determine equilibrium occupancy probabilities were obtained. For a single abnormal trunk in the group, however, a simple exact solution to tbe state equations is possible. This is given in eq. (62). Subsequently, Kaufman[8] has shown that this solution can be generalized to an arbitrary number of abnormal trunks. However, the fraction ineffective and tbe blocking can no longer be expressed in terms of the Erlang B formula, which is the case for one abnormal trunk.

2.1 Terminology

N Number of trunks in group, considered to be numbered from $1, 2, \cdots, N$. (For ordered selection, search for an idle trunk is from $1, 2, \cdots, N$. For two-sided selection, one traffic parcel searches from $1, 2, \cdots, N$, and the other from $N, N-1, \cdots, 1$.)

K Number of the short-holding-time trunk.

λ Arrival rate for the Poisson traffic offered to the group.

μ Hang-up rate for normal trunks, i.e., the holding-time distribution is $F(x) = 1 - e^{-\mu x}$, $x \geqq 0$.

$\tilde{\mu}$ Hang-up rate for short-holding-time trunk.

r Normal- to short-holding-time ratio $(\tilde{\mu}/\mu)$.

a Normal offered load to the group (λ/μ). (For two-sided selection, $a = a_1 + a_2$, where a_1 corresponds to the load offered in direction $1, \cdots, N$ and a_2 the load offered in direction $N, N-1, \cdots, 1$).

P Fraction ineffective (probability that an offered attempt is carried on the short-holding-time trunk).

B Blocking* (probability that an offered attempt finds all trunks busy in the group).

2.2 Analysis for ordered selection†

In this section, we derive results used to determine the effects of an abnormal trunk in a trunk group with an ordered selection of idle trunks. We first derive results for the case where each trunk has a different holding time. We then specialize our results to the case of one abnormal trunk because convenient computational algorithms are available for this case, and because it is the case of most interest.

Of main interest is the proportion of offered calls serviced by each trunk (in equilibrium). Denote these by P_K, $K = 1, 2, \cdots, N$. The P_Ks may be obtained by first calculating the stationary occupany probabilities. If p_K is the stationary occupancy probability of the Kth trunk, then, using (for example) Little's Law,[9] one can show that

$$P_K = \mu_K p_K/\lambda, \tag{1}$$

where μ_K is the hang-up rate for trunk K.

The method used to compute the p_Ks is conceptually straightforward. The interarrival time distribution function of the traffic presented to the Kth trunk is determined, and known results for a single trunk with renewal inputs are used to find p_K.

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* The term *blocking* is used without reference to whether a group is a high usage or final. Blocked attempts for finals are also ineffective attempts.

† This section reproduces and extends some of the results in Ref. 5.

Thus, if the interarrival distrihution function of the traffic into the Kth trunk is $A_K(t)$, and

$$\alpha_K(s) = \int_0^\infty e^{-st} dA_K(t), \tag{2}$$

$$a_K = \int_0^\infty t dA_K(t), \tag{3}$$

$$\rho_K = \frac{1}{a_K \mu_K}; \tag{4}$$

then, using the results in Ref. 10, p. 93,

$$p_K = \rho_K [1 - \alpha_K(\mu_K)]. \tag{5}$$

Hence, the proportion of calls carried hy the Kth trunk is

$$P_K = [1 - \alpha_K(\mu_K)]/a_K \lambda. \tag{6}$$

Following the same argument as in Ref. 10, p. 37, we can ohtain

$$A_K(t) = \int_0^t \{\exp(-\mu_{K-1}x) + [1 - \exp(\mu_{K-1}x)]A_K(t-x)\} dA_{K-1}(x). \tag{7}$$

The idea hehind the argument is to consider that an overflow from the $(K-1)$st trunk occurred at time 0. In order for the next overflow to occur in less than t, we have two cases: an overflow occurs from the $(K-2)$nd trunk at time x, $0 < x < t$ and the $(K-1)$st trunk is husy, or the $(K-1)$st trunk was free at time x, and so the next overflow from the $(K-1)$st trunk must come in less than $t - x$ units of time.

Since the input to the entire trunk group is Poisson,

$$A_1(t) = 1 - e^{-\lambda t} \tag{8}$$

and

$$\alpha_1(s) = \frac{\lambda}{s + \lambda}. \tag{9}$$

Taking the Laplace-Stieltjes transform of (7), we ohtain

$$\alpha_K(s) = \frac{\alpha_{K-1}(s + \mu_{K-1})}{1 - \alpha_{K-1}(s) + \alpha_{K-1}(s + \mu_{K-1})}. \tag{10}$$

Equation (10) can he used to ohtain $\alpha_K(\mu_K)$ for computing P_K via (6). Unfortunately, this computation is quite cumhersome if K is large.

To evaluate a_K, we make use of the fact that

$$a_K = -\frac{d}{ds}\alpha_K(s)\bigg|_{s=0}. \tag{11}$$

After some algebra,

$$a_K = \frac{1}{\alpha_{K-1}(\mu_{K-1})}\, a_{K-1} \tag{12}$$

and, hence,

$$a_K = \frac{1}{\lambda\alpha_1(\mu_1)\alpha_2(\mu_2)\cdots\alpha_{K-1}(\mu_{K-1})}, \tag{13}$$

where $\alpha_j(\mu_j)$ can be obtained from (10). Equations (6) and (13) combine to give the obvious result,

$$P_K = \begin{cases} 1-\alpha_1(\mu_1) & K=1 \\ \alpha_1(\mu_1)\cdots\alpha_{K-1}(\mu_{K-1})(1-\alpha_K(\mu_K)), & K>1, \end{cases} \tag{14}$$

i.e., the probability that an offered attempt is carried on trunk K is the probability it is blocked on the first $K-1$ trunks times the probability it is carried on trunk K, given that it is offered to trunk K. Tbe probability that a call is blocked on the first $K-1$ trunk is simply tbe product of the individual call congestions, i.e., $B_K = B_{K-1}\alpha_K(\mu_K)$, where B_K is the blocking probability on K trunks, with $B_0 = 1$.

In the special case of $\mu_j = \mu$, $j = 1, 2, \cdots, K-1$ and $\mu_K = \tilde{\mu}$, we obtain from (10) and (13)

$$\alpha_K(s) = \frac{1 + \sum_{j=1}^{K-1}\binom{K-1}{j}\lambda^{-j}s(s+\mu)\cdots(s+(j-1)\mu)}{1+\sum_{j=1}^{K}\binom{K}{j}\lambda^{-j}s(s+\mu)\cdots(s+(j-1)\mu)}$$

$$a_K = 1/\lambda B(K-1, a). \tag{15}$$

Here $B(n, a)$ is the Erlang B formula for n trunks offered load a, $(a = \lambda/\mu)$.

In this case, a simple recursion exists for $\alpha_K(s)$. From Ref. 6 we can obtain

$$\alpha_j^{-1}(s) = \frac{s+\lambda+(j-1)\mu}{\lambda} - (j-1)\frac{\mu}{\lambda}\alpha_{j-1}(s) \qquad \text{for} \quad j \leqq K. \tag{16}$$

There is also a simple recursion for $B(j, a)$:

$$B^{-1}(j, a) = 1 + \frac{j}{aB(j-1, a)}, \tag{17}$$

with

$$B(0, a) = 1. \tag{18}$$

Use of recursion (16) with $s = \tilde{\mu}$ together with (17) makes the calculation of the fraction ineffective $P_K = B(K-1, a)[1 - \alpha_K(\tilde{\mu})]$ straigbtforward.

The blocking B (the probability of a call finding all N trunks occupied) can be ohtained by calculating the load overflowing the last trunk. Thus,

$$B = \alpha_N(\mu_N)/\lambda a_N. \tag{19}$$

Even in the case where tbere is only one abnormal trunk in tbe group, this calculation of B can be quite tedious. This is especially true in the case where a large numher of trunks follow the abnormal trunk in the ordering. Instead of (19), we shall approximately calculate B for the special case $\mu_j = \mu$, $j \neq K$.

We consider offering the overflow from trunk K to a hypothetical infinite trunk group with normal bolding times. The mean m and variance v of the number of occupied trunks in the infinite group can be calculated from (this follows from Ch. 3 of Ref. 11, or p. 36 of Ref. 10)

$$m = 1/\mu a_{K+1} \tag{20}$$

$$v = m\left[\frac{1}{1 - \alpha_{K+1}(\mu)} - m\right]. \tag{21}$$

Using (13) and (10), we obtain

$$m = \alpha_K(\tilde{\mu})/a_K\mu = aB(K-1, a)\alpha_K(\tilde{\mu}) \tag{22}$$

$$v = m\left[1 - m + \frac{\alpha_K(\mu + \tilde{\mu})}{1 - \alpha_K(\mu)}\right]. \tag{23}$$

Recursions (16) (used with $s = \mu, \tilde{\mu}, \mu + \tilde{\mu}$) and (17) again make the calculation of m, v straightforward.

We now apply the equivalent random method.[2] That is, we approximate tbe overflow from trunk K by the overflow from an "equivalent" trunk group having normal trunk holding times, and which produces the same m, v on a hypothetical infinite trunk group. To determine blocking, we proceed along the same lines as in the equivalent random method, giving

$$B \approx \lambda_e B\left(N - K + N_e, \frac{\lambda_e}{\mu}\right)\Big/\lambda, \tag{24}$$

where N_e is the number of trunks in the equivalent group and λ_e is the attempt rate for the traffic offered to the equivalent group. (This approximation is exact where one trunk follows trunk K. In other cases, small errors can be expected.)

Althougb the results in this section have assumed that tbe input stream is Poisson, it is a simple matter to extend them to handle

peaked* (overflow) traffic. In particular, one can use the equivalent random method to model the peaked traffic as the overflow from a single trunk group offered Poisson traffic, append this "primary" group to the trunk group of interest, and assume an ordered hunt selection for the extended trunk group. The hunting is done first over the primary group. Thus, if the Kth trunk were faulty in the trunk group of interest and there were N_e trunks in the primary group modeling the peaked traffic, the faulty trunk will now be in the $(N_e + K)$th position.† The proportion of calls carried by the faulty trunk is determined by solving the extended trunk group problem and scaling the resulting proportion up by the ratio of the mean traffic intensity of the input to the extended group to the mean traffic intensity to the original trunk group.

This completes our treatment of ordered selection. Numerical results are given in Section III.

2.3 Analysis for two-sided selection

In this section, two-sided selection is considered. We derive approximations for the fraction ineffective P and for the blocking B.

First, consider the extreme case with trunk K having zero holding time. The fraction ineffective seen by each traffic is $P_1 = B(K-1, a_1)$, $P_2 = B(N-K, a_2)$, where $B(\cdot,\cdot)$ is the Erlang B formula. This case motivates the approximation to be described which, in its simplest form, ignores interaction between the separated subgroups of good trunks, and in more refined form accounts for some interaction.‡ This approximation, which is quite accurate, was used because a computationally feasible solution to the state equations proved difficult.

The notation to be used in this section is indicated in Fig. 1: m_i, v_i are the mean and variance of the traffic offered to trunk K, and c_{ij}, v_{ij} are the mean and variance of the "crossover traffic." To use the convenient recursions developed in the preceding section, only the mean of the crossover traffic is used. This has a negligible effect on the computation for P, but can have a somewhat larger effect on the computation for B, because the peaked crossover traffics would experience a higher blocking than the Poisson traffics. However, their mean is normally small compared to the Poisson traffic, which reduces the effect of their peakedness somewhat. In particular, for $a_1 = a_2$

* The peakedness factor $z(\mu)$ of a traffic stream is the equilibrium variance-to-mean ratio of busy servers when this traffic is offered to an infinite group of exponential servers with service rate μ. The peakedness factor is larger than one for overflow traffic.

† Since nonintegral N_e can occur in the equivalent random method, interpolation may be necessary.

‡ The only real difference is that the refined form requires iteration.

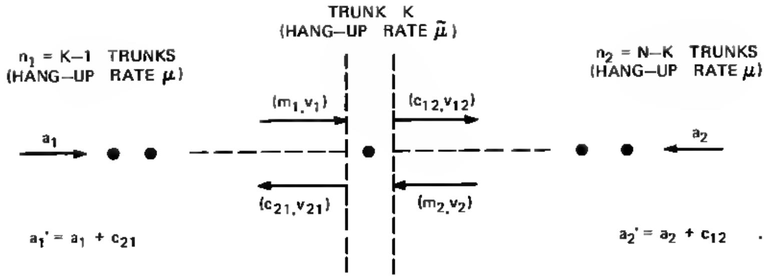

Fig. 1—Terminology for two-sided selection.

(the condition for which the approximation is intended) we usually have $c_{12} \ll a_2$, $c_{21} \ll a_1$. When this is not the case, such as a large trunk group with the short-holding-time trunk in the first position, so that c_{12} may be of the same magnitude as a_2, then c_{12} has a low peakedness.

To describe the computations, assume that the c_{ij} are given and independent of the a_i. Thus, consider a load $a_i' = a_i + c_{ji}$ (treated as Poisson) offered to a group of n_i trunks with hang-up rate μ, overflowing to a trunk with hang-up rate $\tilde{\mu}$. From the results in Section 2.1, the mean and variance of the overflow (in terms of hang-up rate $\tilde{\mu}$) can be determined. First, we define $\lambda_i' = a_i'/\mu$ and solve the recursion [see (16)]

$$\alpha_j^{-1} = \frac{\tilde{\mu} + \lambda_i' + (j-1)\mu}{\lambda_i'} - (j-1)\frac{\mu}{\lambda_i'}\alpha_{j-1}, \tag{25}$$

given

$$\alpha_1 = \frac{\lambda_i'}{\tilde{\mu} + \lambda_i'} \tag{26}$$

for $j = 1, 2, \cdots, n_i + 1$. The parameter $\tilde{\alpha}_i = \alpha_{n_i+1}$ is the call congestion on a (fictitious) trunk with hang-up rate $\tilde{\mu}$. The mean and variance of the offered load to this trunk is

$$m_i' = \frac{1}{r}\, a_i' B(n_i, a_i') \tag{27}$$

$$v_i = m_i'\left(\frac{1}{1 - \tilde{\alpha}_i} - m_i'\right). \tag{28}$$

We must now get the mean and variance of the overflow traffic corresponding to a_i, i.e., for the traffic offered to trunk K. Since c_{ji} is treated as Poisson, clearly

$$m_i = \frac{1}{r}\, a_i B(n_i, a_i'). \tag{29}$$

To get the variance, we use a result from Ref. 12 on the peakedness of split overflow, which gives

$$\frac{v_i}{m_i} = 1 + \frac{a_i}{a_i'}\left(\frac{1}{1 - \bar{\alpha}_i} - m_i' - 1\right). \tag{30}$$

Unless $a_i = a_i'$, this gives a value less than v_i'/m_i'. The total mean and variance of the offered traffic to trunk K is then

$$m = m_1 + m_2 \tag{31}$$

$$v = v_1 + v_2. \tag{32}$$

To compute the call congestion for trunk K, we assume (as in the equivalent random approach) that m, v are the mean and variance of a renewal input to trunk K. If $\alpha(\cdot)$ represents the Laplace-Stieljes transform of the interarrival distribution for this renewal process, then the call congestion is given by $\bar{\alpha} = \alpha(\bar{\mu})$. But for the renewal model,

$$v = m\left(\frac{1}{1 - \bar{\alpha}} - m\right) \tag{33}$$

and, hence,

$$\bar{\alpha} = \frac{m + z - 1}{m + z}, \; (z = v/m). \tag{34}$$

This results in a fraction ineffective for the total offered traffic to the group given by

$$P = \frac{rm}{a_1 + a_2} \cdot \frac{1}{m + z} \tag{35}$$

and an approximation for blocking,

$$B = 1 - P - \frac{a_1'[1 - B(n_1, a_1')] + a_2'[1 - B(n_2, a_2')]}{a_1 + a_2}. \tag{36}$$

Finally, we require that the crossover traffics assumed be consistent with the mean call congestion on trunk K,

$$c_{ij} = a_i B(n_i, a_i') \frac{m + z - 1}{m + z}. \tag{37}$$

This condition can be met by a few simple iteration steps, beginning with $c_{ij} = 0$, and using (37) to update the c_{ij}.

When condition (37) is satisfied, eqs. (35) and (36) give the basic results for the two-sided selection procedure. It should be noted that, in addition to the treatment of each crossover traffic as Poisson and independent of the Poisson traffic offered to the same subgroup of good trunks, a second (minor) approximation is implicit in the computations. This is the use of the average call congestion in (37) to update

each crossover traffic. It is a straightforward matter to determine a separate call congestion for each traffic offered to trunk K, but the approximation is sufficiently accurate without this refinement. Numerical results using the approximations and comparisons with exact solutions (for small problems) are presented in Section III.

2.4 Analysis for queuing and random selection

To derive results for the case where the idle trunks form a queue, we define a two-dimensional state that represents the number of idle trunks in the queue together with the position of the abnormal trunk in the queue. We then derive equations for the equilibrium probabilities of each state. One can show that these probabilities exist and are unique.* The equilibrium probability that the abnormal trunk is busy, denoted by $\tilde{p}$, is then simply a sum for states in which the abnormal trunk is busy. As indicated in Section 2.2, it is easy to show that the mean number of requests serviced by the abnormal trunk in a unit of time is given by

$$\tilde{\mu}\tilde{p} \tag{38}$$

and, hence, the proportion of requests serviced by the abnormal trunk is

$$P = \tilde{\mu}\tilde{p}/\lambda = r\tilde{p}/a. \tag{39}$$

In the course of deriving our results, we show that it is equally likely that the abnormal trunk occupies any position in the queue of idle trunks. This, however, is equivalent to a random selection of idle trunks as far as the fraction ineffective and blocking are concerned. Hence, the results derived for P, B also apply to the random selection model. Strictly speaking, the two selection procedures are not equivalent, of course. For example, with queuing selection, an abnormal trunk would not serve two successive attempts if other trunks were idle, but this is a possibility with random selection. Under other assumptions, e.g., dependent retrials or non-Poisson input, these differences become important.

To proceed with the derivation, define

E_{ij} = the event that there are j idle trunks in the queue and that the abnormal trunk is in the ith position in the queue.

P_{ij} = Prob $\{E_{ij}\}$.

We let $i = 0$ denote that the abnormal trunk is occupied. Thus,

$$\tilde{p} = \sum_{j=0}^{N-1} P_{0j}. \tag{40}$$

* We have a continuous-time Markov process, with a well-behaved embedded Markov chain (see Ref. 13).

The following equations can be readily derived relating the P_{ij}'s:

$$[\tilde{\mu} + (N-1)\mu]P_{00} = \lambda P_{01} + \lambda P_{11}, \tag{41}$$

$$[\tilde{\mu} + (N-j-1)\mu + \lambda]P_{0j} = \lambda P_{0,j+1} + (N-j)\mu P_{0,j-1} + \lambda P_{1,j+1} \quad \text{for} \quad 1 \leqq j < N-1, \tag{42}$$

$$[\tilde{\mu} + \lambda]P_{0,N-1} = \lambda P_{1N} + \mu P_{0,N-2}, \tag{43}$$

$$[(N-j)\mu + \lambda]P_{ij} = (N-j+1)\mu P_{i,j-1} + \lambda P_{i+1,j+1} \quad \text{for} \quad 0 < i < j \leqq N, \tag{44}$$

$$[(N-j)\mu + \lambda]P_{jj} = \tilde{\mu} P_{0,j-1} + \lambda P_{j+1,j+1} \quad \text{for} \quad 0 < j \leqq N, \tag{45}$$

and, trivially,

$$\begin{aligned} P_{ij} &= 0 \quad \text{for} \quad i > j, \quad \text{or} \quad j > N; \\ P_{0j} &= 0 \quad \text{for} \quad j \geqq N. \end{aligned}$$

These equations appear quite formidable because of the apparent lack of simple structure. A brute force algebraic approach seems unfruitful, as does a generating function approach.

Instead, we make the conjecture that

$$P_{1j} = P_{2j} = \cdots = P_{jj} \qquad \text{for all } j \geqq 1. \tag{46}$$

We should note that (46) is equivalent to assuming that the idle trunk is selected at random. This is true because P_{ij} represents the probability of seeing the state (i, j) at a random instant in equilibrium and Poisson arrivals see the same distribution.

We justify our conjecture by using (46) in (41) to (45) together with the fact that the probabilities must add to 1 and showing that the resulting equations have a simple solution. This solution can be substituted into the original equations to show that we indeed have the correct solution. However, since there exists a unique equilibrium solution to these equations and we have, in effect, solved them by adding additional constraints, the proof is complete without this step.

Proceeding in the manner described, we first use (46) in (41) to (45) and obtain the following equations:

$$[\tilde{\mu} + (N-1)\mu]P_{00} = \lambda P_{01} + \lambda P_{11}, \tag{47}$$

$$[\tilde{\mu} + (N-j-1)\mu + \lambda]P_{0j} = \lambda P_{0,j+1} + (N-j)\mu P_{0,j-1} + \lambda P_{j+1,j+1} \quad \text{for} \quad 1 \leqq j < N-1, \tag{48}$$

$$[\tilde{\mu} + \lambda]P_{0,N-1} = \lambda P_{NN} + \mu P_{0,N-2}, \tag{49}$$

$$[(N-j)\mu + \lambda]P_{jj} = (N-j+1)\mu P_{j-1,j-1} + \lambda P_{j+1,j+1} \quad \text{for} \quad 0 < j \leqq N, \tag{50}$$

and

$$[(N - j)\mu + \lambda]P_{jj} = \tilde{\mu}P_{0,j-1} + \lambda P_{j+1,j+1} \quad \text{for} \quad 0 < j \leqq N. \tag{51}$$

Putting $j = N$ in eq. (50), recalling that $P_{N+1,N+1} = 0$, we get

$$P_{N-1,N-1} = \frac{\lambda}{\mu} P_{NN} = aP_{NN}. \tag{52}$$

Repeatedly reducing j by 1 and solving for $P_{j-1,j-1}$ in terms of P_{NN}, we get

$$P_{jj} = \frac{a^{N-j}}{(N-j)!} P_{NN}, \qquad j = 1, \cdots, N. \tag{53}$$

Using (53) in (51) results in

$$P_{0j} = \frac{\lambda}{\tilde{\mu}} \frac{a^{N-j-1}}{(N-j-1)!} P_{NN}. \tag{54}$$

Finally, from (47) we get

$$P_{00} = \frac{\lambda}{\tilde{\mu}} \frac{a^{N-1}}{(N-1)!} P_{NN}. \tag{55}$$

One can check that expressions (53), (54), and (55) satisfy eqs. (48) and (49), which proves the conjecture.

We compute P_{NN} from the fact that the sum of the probabilities must be 1. This yields

$$P_{NN} = \frac{1}{N}\left[\frac{a^N}{N!}\frac{\mu}{\tilde{\mu}} + \cdots + \frac{a^j}{j!}\left(\frac{N-j}{N} + \frac{j}{N}\frac{\mu}{\tilde{\mu}}\right) + \cdots + 1\right]. \tag{56}$$

Thus,

$$\tilde{p} = \sum_{j=0}^{N-1} P_{0j} = \frac{\mu a \sum_{j=0}^{N-1} \frac{a^j}{j!}}{\tilde{\mu} N \sum_{j=0}^{N} \frac{a^j}{j!}\left(\frac{N-1}{N} + \frac{j\mu}{\tilde{\mu}}\right)} \tag{57}$$

and

$$P = \frac{r\tilde{p}}{a} = \frac{\sum_{j=0}^{N-1} \frac{a^j}{j!}}{N \sum_{j=0}^{N} \frac{a^j}{j!}\left(\frac{N-j}{N} + \frac{j}{rN}\right)}, \tag{58}$$

recalling that $r = \tilde{\mu}/\mu$.

Equation (58) can be rewritten in terms of Erlang's loss function as

$$P = \frac{\sum_{j=0}^{N-1} \frac{a^j}{j!}}{N \sum_{j=0}^{N} \frac{a^j}{j!} + \left(\frac{1}{r} - 1\right) \sum_{j=1}^{N} \frac{a^j}{(j-1)!}}$$

$$= \frac{\sum_{j=0}^{N-1} \frac{a^j}{j!}}{N \sum_{j=0}^{N-1} \frac{a^j}{j!} + \frac{Na^N}{N!} + \left(\frac{1}{r} - 1\right) a \sum_{j=0}^{N-1} \frac{a^j}{j!}}.$$

Thus,

$$P = \frac{r}{a + Nr - ar[1 - B(N-1, a)]}. \tag{59}$$

An alternative form of (59) can be obtained via the standard recursion for Erlang B [see (17)], giving

$$P = \frac{r[1 - B(N, a)]}{Nr - (r-1)a[1 - B(N, a)]}. \tag{60}$$

To get an expression for the blocking B, notice that

$$B = P_{00} = \frac{\lambda}{\bar{\mu}} \frac{a^{N-1}}{(N-1)!} P_{NN} = \frac{(\lambda/\bar{\mu})(a^{N-1}/(N-1)!)}{N\left[\sum_{j=0}^{N} \frac{a^j}{j!}\left(\frac{N-j}{N} + \frac{j}{rN}\right)\right]}$$

or

$$B = \frac{NB(N, a)}{Nr - (r-1)a[1 - B(N, a)]}. \tag{61}$$

Although (61) and (62) also apply to the random model, it should be noted that the equivalence between random and queuing selection extends to the equilibrium distribution for *redefined* states (i, j), where i is the number of good trunks occupied and j is 1 if the abnormal trunk is occupied, 0 otherwise. In fact, it is straightforward to show from (53) to (55) that the redefined probabilities must satisfy

$$\left.\begin{aligned} P_{i0} &= \frac{a^i(N-i)}{i!\,N} P_{00} \\ P_{i1} &= \frac{a^{i+1}}{ri!\,N} P_{00} \end{aligned}\right\} i = 0, 1, \cdots, N-1, \tag{62}$$

which solve the somewhat simpler equations for the random model (these equations are given in Ref. 4).

III. NUMERICAL RESULTS

This section presents selected numerical results obtained with the solution procedures of Section II.

3.1 Ordered selection (Figs. 2 to 5)

Figure 2 gives the fraction ineffective for $r = 5$ and the blocking* for $r=5$, $N=20$. Even for this relatively small value of r, the short-holding-time trunk has a significant impact, which decreases as the position in the order of selection increases. For large values of offered load, the dependence of fraction ineffective on K (position of the short-holding-time trunk) is decreased. In fact, it follows from Little's Law[9] that, for all K (and all selection procedures), $P \leqq r/a$, with the bound approached as the time congestion on the short-holding-time trunk approaches 1.0; i.e., for very large a. For $a = 25$, $K = 1$, we note that $P = 0.17$, $r/a = 0.2$.

The results for fraction ineffective do not depend on the total number of trunks $N \geqq K$. For blocking, N is important. For $N = 20$, $r = 5$, blocking is decreased from Erlang B blocking by about $\frac{1}{3}$ in the (design) range for 20 to 30 percent overflow, with the decrease relatively insensitive to K. Blocking results are approximate, with errors resulting only from the application of the equivalent random method. In cases where exact solutions were compared to the approximate, the agreement was good. The decreased overflow resulting from a short-holding-time trunk can contribute substantial errors in estimation procedures for which overflow enters. Usage measurements are also affected by a short-holding-time trunk, with mean carried usage given by $aP/r + a(1 - P - B)$.

Figure 3, for $r = 15$, is similar to Fig. 2, but with larger impacts and more spread between the curves. The ratio $r = 15$ is typical for a short-holding-time trunk and shows that $P > 0.4$ can easily occur, even for relatively large loads. The limiting ratio $r = \infty$ (zero-holding-time trunk) would result in $P^{\infty} = B(K - 1, a)$, and $B^{\infty} = 0$, which are substantially different from the $r = 15$ results. For example, $P^{\infty} = 1$ for $K = 1$.

3.2 Two-sided selection (Figs. 4 and 5)

Figure 4 gives the behavior for $N = 5$, $r = 15$, including a comparison with the exact solution obtained by solving the state equations for the system. The approximation for P displays essentially no error for low blocking. This is to be expected for conditions under which the

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* Recall that blocking refers to the fraction of attempts that find all trunks busy in the group, whether or not it is a final or high-usage group.

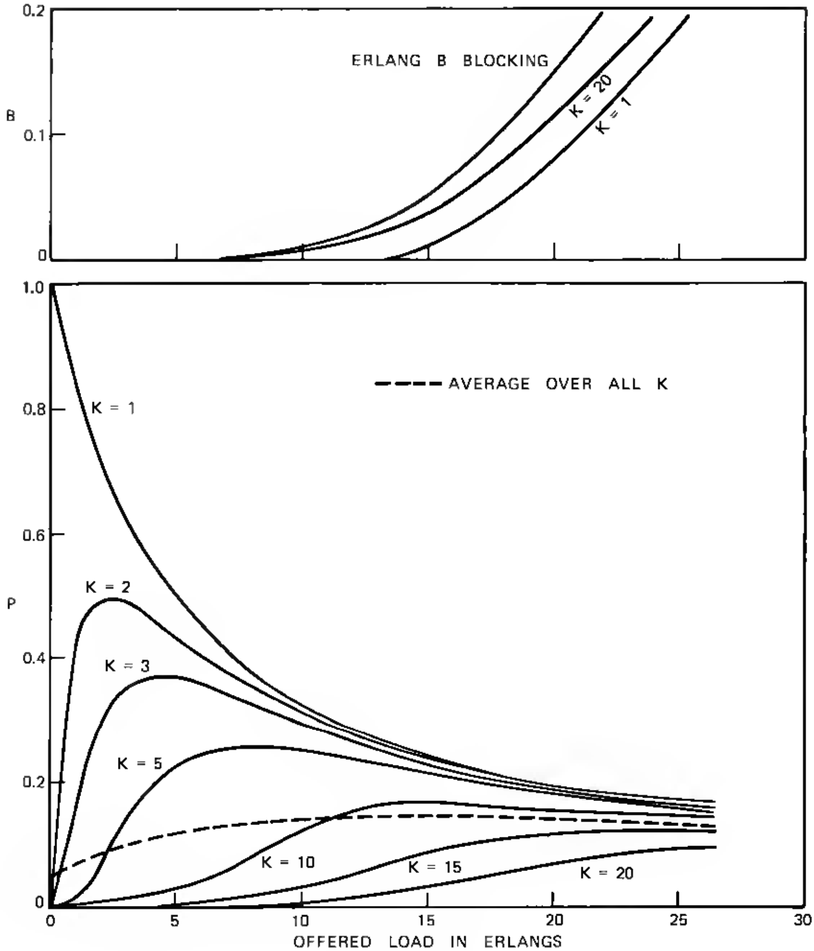

Fig. 2—Behavior of ordered selection for $r = 5$, $N = 20$.

offered load to trunk K comes from only one side (e.g., $K = 1$, low offered load), since the approximation for P becomes exact. For other low blocking conditions (such as $K = 3$, low offered load), no discernible error indicates that modeling the traffics offered to trunk K as a renewal stream introduces essentially no error in the computed call congestion for trunk K. As the crossover traffics increase, and, hence, there is interaction between the separated subgroups of good trunks, the approximation shows some error. The error is well-behaved, and increases to only about 2 to 3 percent. For larger groups, the error is generally less and is not shown on the figures.

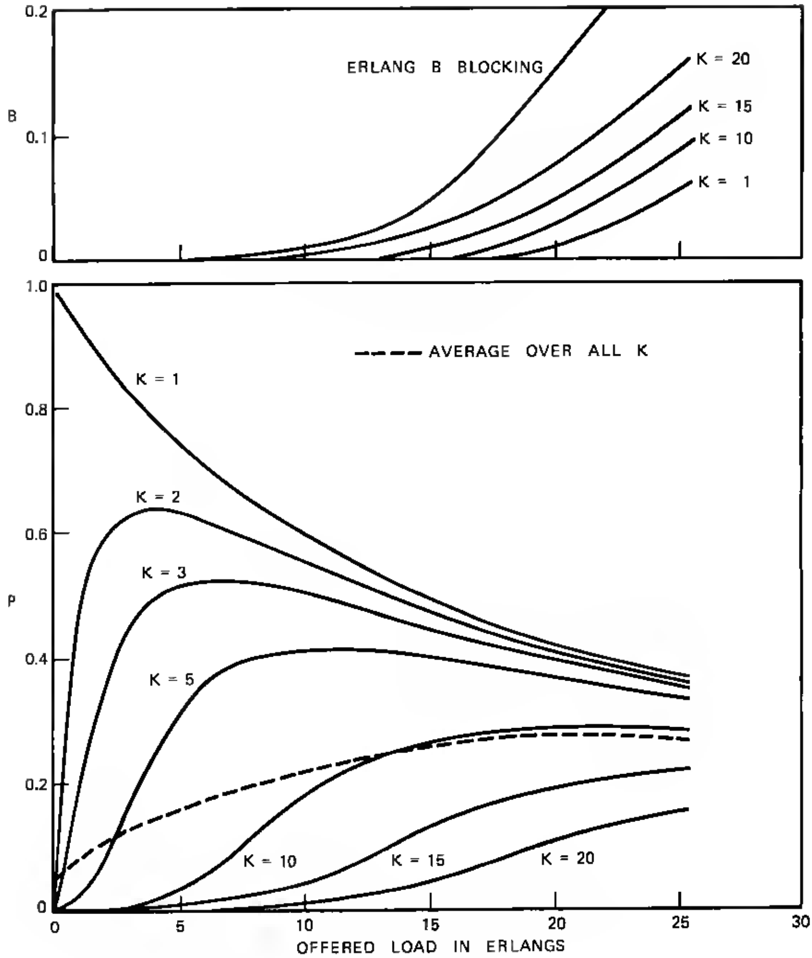

Fig. 3—Behavior of ordered selection for $r = 15$, $N = 20$.

Errors in the blocking are larger. Since peakedness is ignored in determinining the crossover traffics and the carried load on the good trunks, this is to be expected. However, the absolute error does not increase significantly with offered load. As the change from Erlang B blocking is the important factor in applications, this error behavior is acceptable. As for ordered selection, the blocking is relatively insensitive to K.

The interesting behavior for $K = 1$ is apparently due to the size of the group. For $K = 1$, $N \geqq 2$, qualitatively, one would expect that the fraction ineffective seen by load a_1 monotonically decreases from

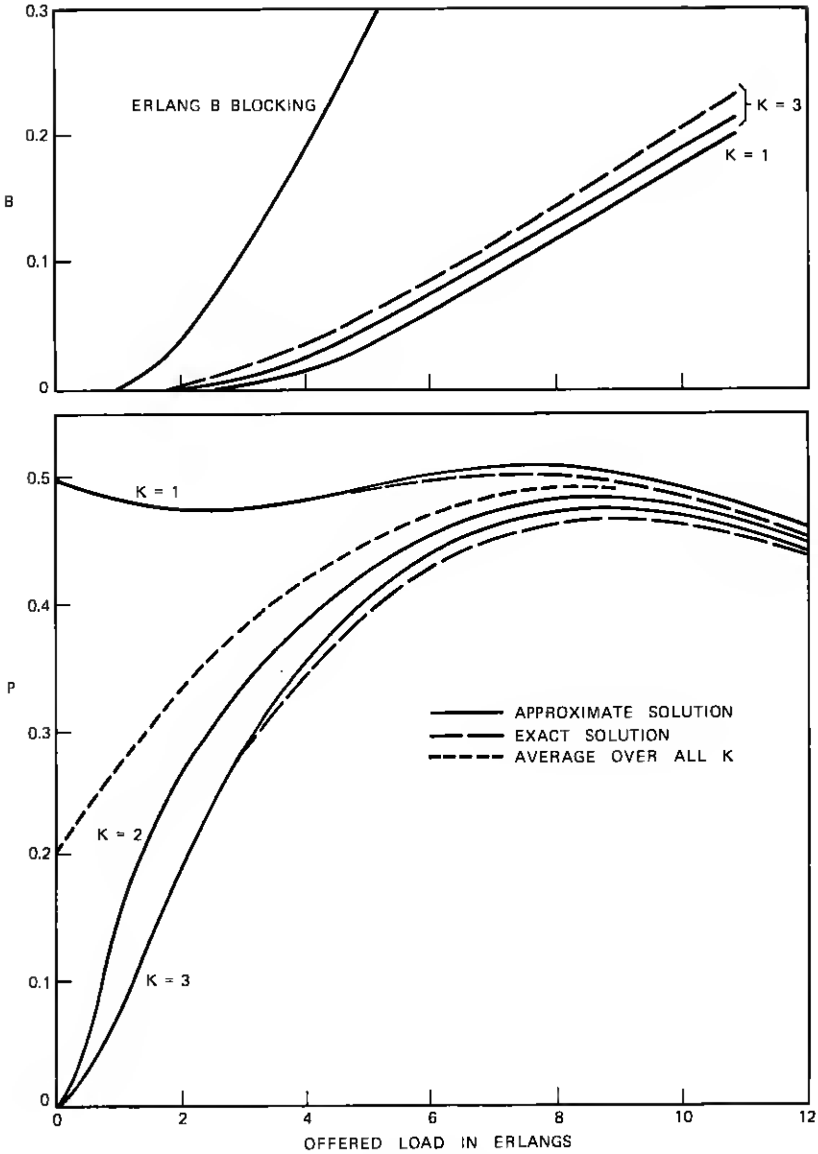

Fig. 4—Behavior of two-sided selection for $N = 5$, $r = 15$.

1.0, while for a_2 it initially increases from 0, as the offered load increases. The exact solution for $N = 5$ confirmed this behavior. However, the fraction ineffective seen by load a_2 increases fast enough to create an increase in P, before its ultimate decrease. As a result,

$P \approx 0.5$ in the relatively wide range 0 to 10 erlangs. For larger values of N, P was found to be monotonically decreasing for $K = 1$. On the other hand, for $N = 2$, P can easily be shown to initially increase. Since $a_1 = a_2$, for $N = 2$, two-sided and random selection are equivalent, and (59) applies. Thus, the conditions that produce the qualitative behavior of Fig. 4 are difficult to predict.

Figure 5 illustrates the behavior for $r = 15$, $N = 20$. Compared to ordered selection, the impact is less, and the dependence on K less pronounced, especially for blocking. This is attributable to the diversity introduced into the selection procedure by two-sided selection. However, a substantial overall impact is still evident. For $N = 20$, even for the most favorable case $K = 10$, for a design overflow of 20 percent, $P \approx 0.2$, and overflow is reduced to 5 to 6 percent.

The limiting case for $r = \infty$ gives

$$P^\infty = \frac{B(K-1, a/2) + B(N-K, a/2)}{2}.$$

For $N = 20$, $K = 1$, this gives a value of 0.5 over the range 0 to 25 erlangs, compared to the value of $P = 0.29$ for $a = 25$, $r = 15$. For $N = 20$, $K = 10$, $P^\infty = 0.35$ at $a = 25$, or about 50 percent higher than the value for $r = 15$.

3.3 Random/queuing selection (Figs. 6 and 7)

In each case in Figs. 6 and 7, P begins at $1/N$, then increases over the range of offered loads shown. The $1/N$ initial behavior is intuitively obvious for random selection. Queuing selection, on the other hand, guarantees that, when a trunk becomes idle, trunks already idle must serve calls before it can be picked for service. For very low offered loads, a short-holding-time trunk essentially serves every Nth call to give the $1/N$ initial behavior.

Analysis of eq. (59) indicates that P can have at most one extremum. For $r > 1$, this occurs at the unique root of the equation

$$\frac{\partial}{\partial a}\{a[1 - B(N-1, a)]\} = \frac{1}{r}.$$

Thus, random/queuing selection never displays the complex behavior observed in Fig. 4 for two-sided selection. Because the extremum occurs for relatively large values of a, a short-holding-time trunk has a larger impact in a high usage group than in a final group of the *same* size. For example, for ten trunks, 4.5 erlangs is a typical offered load for a final, and 9.5 erlangs for a high usage group. From Fig. 6, $P = 0.17$ at $a = 4.5$ erlangs, whereas $P = 0.28$ at $a = 9.5$ erlangs, i.e., the *relative* impact is larger. The time congestion aP/r for the short-holding-

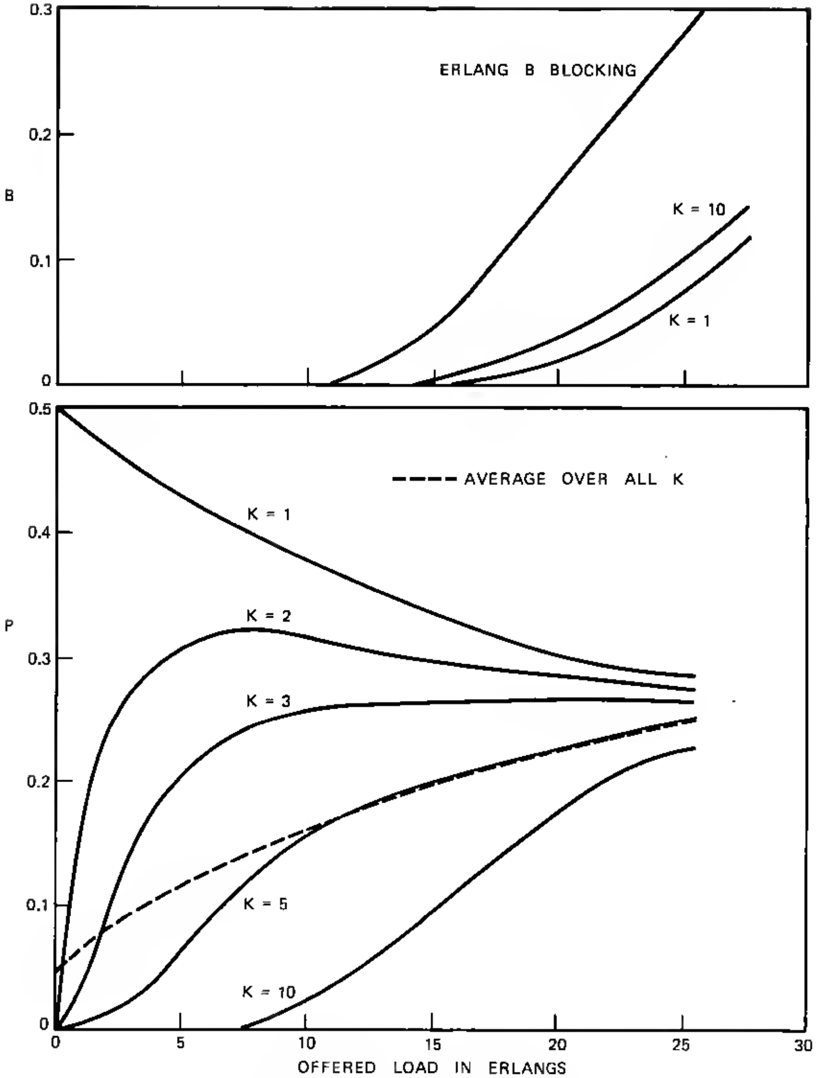

Fig. 5—Behavior of two-sided selection for $N = 20$, $r = 15$.

time trunk is 0.052 at $a = 4.5$ erlangs, and 0.175 at $a = 9.5$ erlangs, i.e., the *absolute* impact of the short-holding-time trunk is over three times as much for the high-usage application. In terms of the expected time congestion with a short-holding-time trunk equally likely in any position, this general behavior also holds for two-sided and ordered selection. This follows from the increase in average fraction ineffective as the offered load increases.

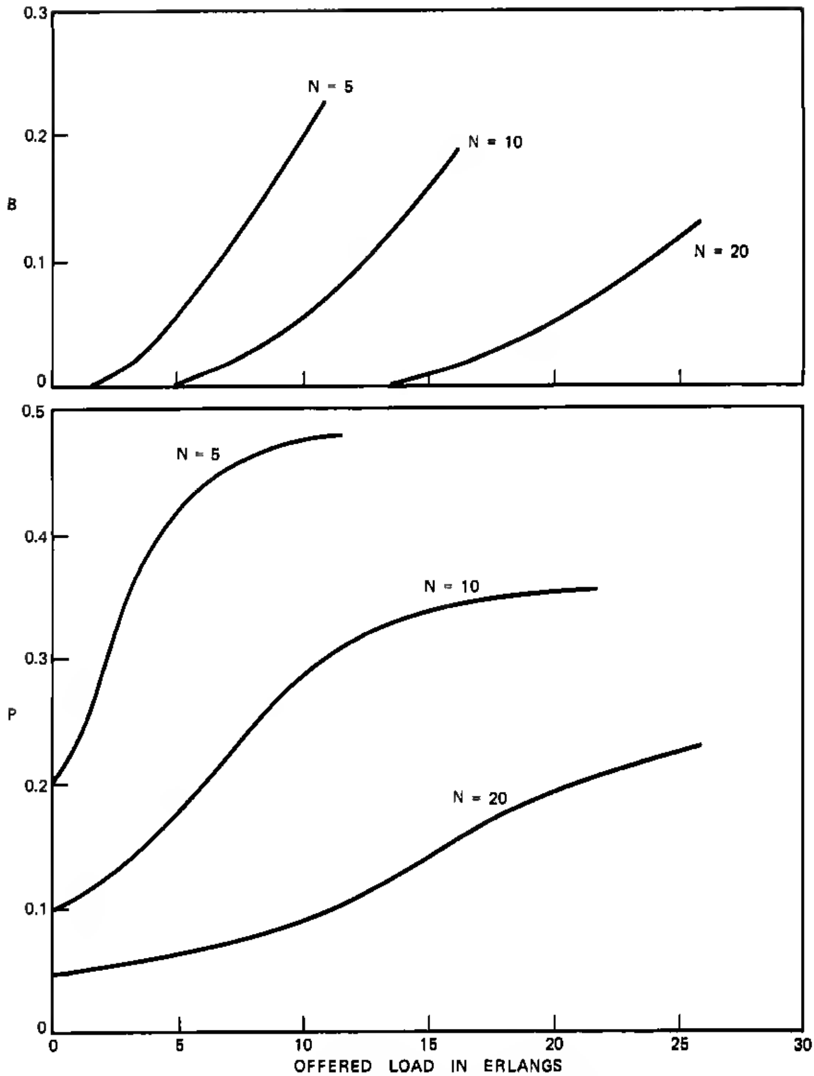

Fig. 6—Behavior of random selection for $r = 15$, $N = 5, 10, 20$.

Figure 7 displays the behavior of random/queuing selection as r varies. In all cases, P is close to its maximum over a fairly wide range. The limiting case $r = \infty$ gives [see (59)]

$$P^{\infty} = \frac{1}{N - a[1 - B(N-1, a)]}.$$

For $N = 20$, $a = 25$ erlangs, $P^{\infty} = 0.36$, compared to the $r = 20$

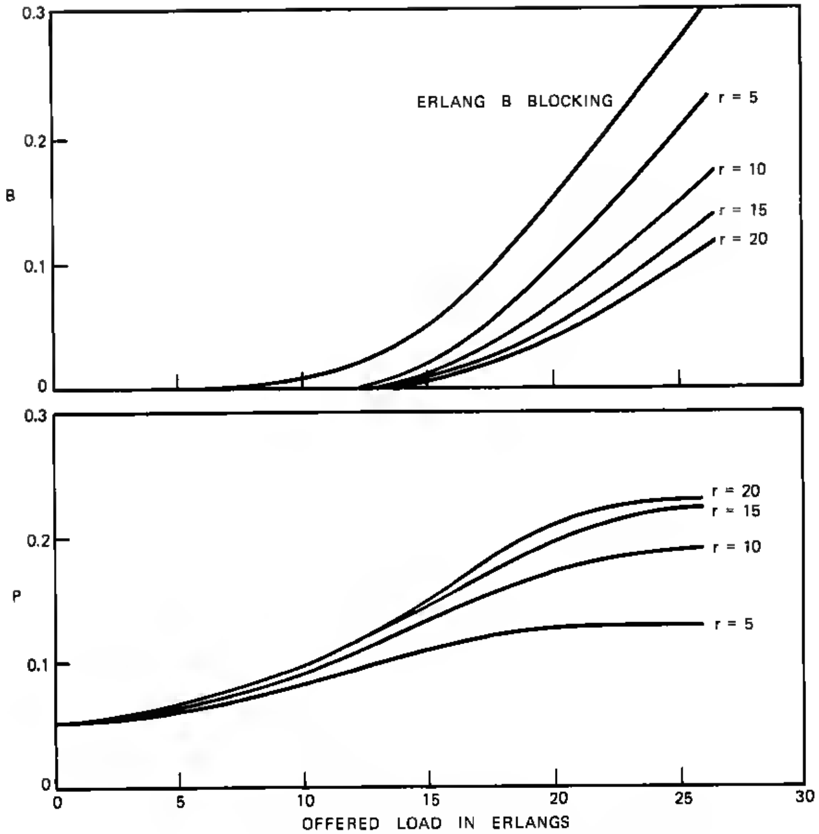

Fig. 7—Behavior of random selection for $N = 20$, $r = 5, 10, 15, 20$.

value from Fig. 7 of $P = 0.23$. Since P is no more difficult to compute than P^{∞}, the upper bound P^{∞} is of limited use in this case.

3.4 Comparison of selection procedures

To compare selection procedures, the measure taken is the expected value of P, denoted by $\bar{P}$, given that a short-holding-time trunk is equally likely in any position. Computationally, it was found that

$$\bar{P}\begin{Bmatrix}\text{Random}\\\text{or}\\\text{queuing}\\\text{selection}\end{Bmatrix} \leqq \bar{P}\begin{bmatrix}\text{Two-sided}\\\text{selection}\end{bmatrix} \leqq \bar{P}\begin{Bmatrix}\text{Ordered}\\\text{selection}\end{Bmatrix},$$

with equality occurring only at $a = 0$. Figure 8 for $N = 20$, $r = 15$ displays typical behavior. The differences are substantial. It is likely that

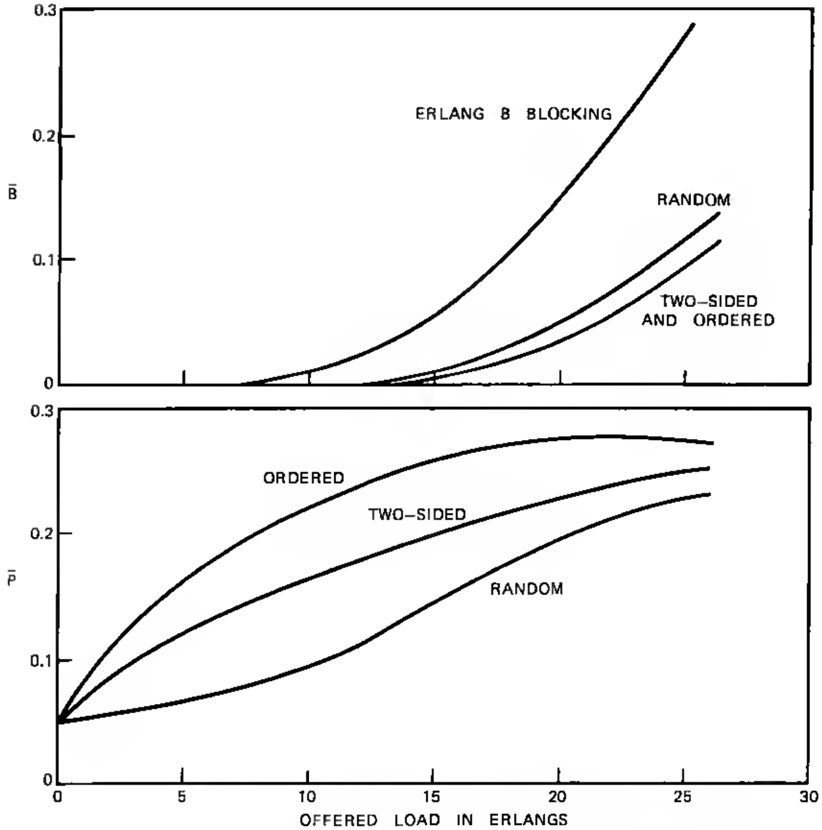

Fig. 8—Comparison of selection procedures for short-holding-time trunk equally likely in any position ($N = 20$, $r = 15$).

any improvement over the random/queuing behavior would require a selection procedure based on more information. For example, choosing the idle trunk wbose last holding time was longest would further reduce tbe impact of a short-holding-time trunk.

The expected blocking does not show much difference from one selection procedure to another, but all disciplines show a substantial reduction from tbe Erlang B results.

3.5 Validation remarks

The basic traffic assumptions of the model concern bolding times and the arrival process. For normal trunk bolding times (which result from a mix of conversations, busys, don't answers), the exponential assumption is reasonable. For short-holding-time trunks, this assumption may be less valid. For example, trunks resulting in immediate

reorder returned to the customer would not likely display tbe exponential behavior. However, since the Erlang B formula holds for arbitrary holding-time distributions, it is reasonable to expect that the results should not be too sensitive to the form of the distribution.

The Poisson assumption for the arrival process can be invalidated due to time variations, to peakedness (for groups receiving overflow traffic), or to retrial behavior. For ordered selection with $K = 1$, peakedness clearly reduces the impact of a short-holding-time trunk. This behavior may be reversed for higher values of K since peakedness increases the mean attempt rate to trunk K. For example, for $r = \infty$, $K \geq 2$, P is increased. However, several peaked traffic computations for $r = 15$ give values of the fraction ineffective that are very close to the Poisson value.

Even if the Poisson assumption is valid with all trunks good, it can be violated in the presence of a short-holding-time trunk. Tbe short time between a retrial and tbe failure that initiated it gives the retrial a high probability of meeting the same conditions as tbe initial failure. For two-sided, ordered, and random selection, this could increase the impact of the short-holding trunk over what would be expected from a Poisson assumption.

The model is also susceptihle to variations from the idealized selection procedures. This holds particularly for applications to older systems where grading and multipling arrangements can lead to many different selection procedures. It also holds for applications to 5XB systems, where any irregularity in the distrihution of trunks on the frames can distort the selection procedure from the idealized random selection model. In fact, individual circuit usage results would seem to indicate that significant discrepancies from (cyclic) random selection may be quite common in 5XB. Thus, for any specific application, suitability of the model would have to he determined.

Figure 9 shows data obtained hy special measurements from a Crossbar Tandem group containing a short-holding-time trunk. The peg count and usage measurements during the study hour indicate that trunk 2 had a short holding time relative to other trunks in the group, and a failure condition was suhsequently verified. Thus, all 35 attempts on the trunk are assumed to have failed. The large imbalance in the offered attempts from either side of the group could be statistical, or it could be due to irregularities in incoming trunk assignment to frames (which determines the order of selection). It is more likely due to retrials. Thus, the reasonable assumption that $a_1 = a_2$ can be violated in some situations.

Since the measured hour was not the husy hour, tbe left side of the group can be treated like ordered selection. The estimate for a mean

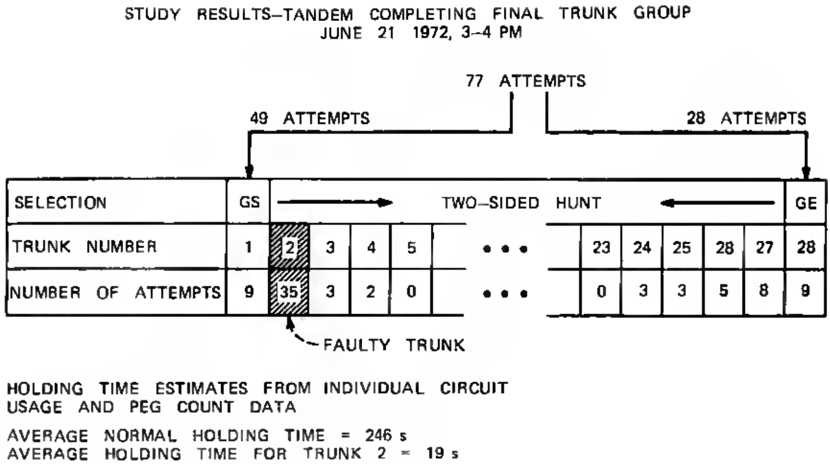

Fig. 9—Example of a short-holding-time trunk.

attempt rate for a Poisson offered traffic is $\hat{\lambda} = 49$ attempts/hour. For the offered load a in erlangs, a normal holding time is associated with each attempt, to give $\hat{a} = 3.3$ erlangs. The ratio of holding times is $r = 13$. For these values, the ordered selection model with $K = 2$ predicts $P = 0.62$. Thus, the realized value of ineffectives (35) is close to the average value (30.5) predicted from the same data. The discrepancy could be statistical, or to modeling the stream as Poisson, which smoothes out the retrial stream.

The preceding very limited comparison is not a validation. Because of the various factors noted, a comprehensive validation of the models is difficult. Since the models are not intended for design purposes, but only to estimate impact of short-holding-time trunks, such validation has not been attempted.

IV. CONCLUSIONS

This paper has considered analytical models for groups containing short-holding-time trunks. The models confirm that tbese trunks can have a substantial impact on customer service. Although traffic and system characteristics can differ from those of the model, it is felt that the models are adequate for estimation purposes.

The numerical results indicate that the type of system and type of group (high usage or final) lead to significant differences in performance in the presence of a short-holding-time trunk. These models and results are directly useful in devising optimum strategies for deploying maintenance resources to minimize the service impact of short-holding-time trunks.

V. ACKNOWLEDGMENTS

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